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The Field of a Point Source in an Elastic Homogeneous Anisotropic Medium

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Abstract—A new method is applied to obtain the Green's tensor for a point source in a general anisotropic elastic homogeneous unbounded medium in the form of an integral over the surface of a unit sphere. This representation can be used to obtain, in the time domain, simple expressions for the displacement field generated by a localized source with an arbitrary time dependence. The static term of the Green's tensor is transformed into an easily calculable form. A high-frequency asymptotic expression is derived for the Green's tensor.

1. INTRODUCTION

Seismological studies show that transversal and even orthorhombic anisotropy is characteristic, to varying degrees, of all types of rock, especially sedimentary and metamorphic. In most cases, the anisotropy (including thin stratification) is presumed to be caused by oriented fracturing [1–3]. Knowledge of the orientation and intensity of fracturing, evaluated by seismic-sounding methods, is very important at all stages of exploration and exploitation of oil and gas fields. Correct solution of this sounding problem entails numerous consequences, especially concerning reservoir rocks, where fracturing determines filtration properties [4]. Moreover, seismic-sounding of fracture systems is required for a variety of other geological and technological purposes [1, 2].

Thus, analysis of wave propagation from point sources in anisotropic media is of practical consequence. In the case of an unbounded homogeneous medium, special emphasis is placed on construction of the Green's tensor. In distinction to [5–8], this paper presents a new derivation of an integral representation for the Green's tensor, which provides a simple procedure for expressing the displacement field generated by a localized source with an arbitrary time dependence. We show that the static term can be reduced to a simple integral, easily calculated by the residue theorem. Applying the stationary phase method, we find the leading term of an asymptotic expression for the obtained solution in the case of nondegenerate stationary points. It should be noted that the derivation [6] of an expression for the displacement, induced by a point source with an arbitrary time dependence, contained an error, and the final expression {see equation (C20) in [6]} did not include the static term [i.e., the first term in equation (29) of the present paper]. Applying the Radon transformation to the steady-state case, which was done in [5, 8], leads to an expression identical to expression (26) of this paper.

2. DERIVATION OF AN INTEGRAL REPRESENTATION FOR THE GREEN'S TENSOR

In mathematical terms, the derivation of an integral representation of the Green's tensor problem reduces to the set of differential equations

$$\partial_t^2 u_i - \lambda_{ijkl}\partial_j\partial_k u_l = F(\mathbf{x}, t)e_i, \quad (i, j, k, l = 1, 2, 3), \tag{1}$$

where $u_i = u_i(\mathbf{x}, t)$, $\partial_k = \frac{\partial}{\partial x_k}$, $\partial_t^2 = \frac{\partial^2}{\partial t^2}$, λ_{ijkl} are the elements of the tensor of elastic coefficients normalized to density; and $\mathbf{x}^T = (x_1, x_2, x_3)$, where $\mathbf{x}^T$ is a row matrix, $\mathbf{x}$ is a column matrix, and T indicates a transpose. The required solution obeys the conditions

$$u_i(\mathbf{x}, t) \equiv 0 \quad \text{for} \quad t < 0. \tag{2}$$

It is convenient to use the Hamiltonian operator $\nabla = (\partial_1, \partial_2, \partial_3)^T$ to introduce the matrix

$$\Lambda(\nabla) = \|\Lambda_{il}(\nabla)\| = \|\lambda_{ijkl}\partial_j\partial_k\| \tag{3}$$

and rewrite equation (1) in the matrix form as

$$[\partial_t^2\mathbf{I} - \Lambda(\nabla)]\mathbf{U} = F(\mathbf{x}, t)\mathbf{e}, \tag{4}$$

where $\mathbf{U}^T = (u_1, u_2, u_3)$, $\mathbf{e}^T = (e_1, e_2, e_3)$, and $\mathbf{I}$ is the three-dimensional identity matrix.

We assume that $F(\mathbf{x}, t)$ is a delta-type source:

$$F(\mathbf{x}, t) = f(t)\frac{\varepsilon}{\pi^2(|\mathbf{x}|^2 + \varepsilon^2)^2}, \quad \varepsilon > 0, \tag{5}$$

where $f(t)$ is an arbitrary function such that $f(t) \equiv 0$ at $t < 0$. Here,

$$\frac{\varepsilon}{\pi^2(|\mathbf{x}|^2 + \varepsilon^2)^2} \longrightarrow \delta(\mathbf{x}) \text{ for } \varepsilon \longrightarrow 0, \tag{6}$$

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and $\delta(\mathbf{x})$ is the three-dimensional Dirac delta function. Relation (6) is proved by taking the Fourier transform of the source function:

$$\int_{R^3} \exp[-i\mathbf{kx}]\frac{\varepsilon}{\pi^2(|\mathbf{x}|^2+\varepsilon^2)^2}d\mathbf{x} = \exp[-\varepsilon|\mathbf{k}|]. \quad (7)$$

Now we take the Fourier transforms of both sides of equation (4) with respect to time and spatial coordinates and use (7) to obtain

$$[\Lambda(\mathbf{k}) - \omega^2\mathbf{I}]\mathbf{U}(\mathbf{k}, \omega) = \exp[-\varepsilon|\mathbf{k}|]f(\omega)\mathbf{e}. \quad (8)$$

Here,

$$\mathbf{U}(\mathbf{k}, \omega) = \int_0^\infty \exp[i\omega t]dt \int_{R^3} \mathbf{U}(\mathbf{x}, t)\exp[-i\mathbf{kx}]d\mathbf{x}, \quad (9)$$

where $\operatorname{Im}\omega > 0$. In this case, the inverse Fourier transform with respect to ω is

$$\mathbf{U}(\mathbf{x}, t) = \frac{1}{2\pi}\int_{-\infty+i0}^{+\infty+i0} \mathbf{U}(\mathbf{x}, \omega)\exp[-i\omega t]d\omega, \quad (10)$$

which ensures that condition (2) holds. For the sake of brevity, we use $\mathbf{U}$ to denote different functions, indicating their respective domains by the argument.

Expressing $[\Lambda(\mathbf{k}) - \omega^2\mathbf{I}]$ as a power series in terms of eigenvectors, we write the solution to (8) in the form

$$\mathbf{U}(\mathbf{k}, \omega) = f(\omega)\exp[-\varepsilon|\mathbf{k}|]\sum_{j=1}^{3}\frac{\mathbf{A}_j(\mathbf{n})\mathbf{A}_j^T(\mathbf{n})}{\mathbf{k}^2\nu_j^2(\mathbf{n}) - \omega^2}\mathbf{e}, \quad (11)$$

where we used the relations $\nu(\mathbf{k}) = |\mathbf{k}|\nu(\mathbf{n})$ and $\mathbf{n} = \mathbf{k}/|\mathbf{k}|$; $\mathbf{A}_j(\mathbf{n})$ is the eigenvector of the matrix $\Lambda(\mathbf{n})$ associated with the eigenvalue $\nu_j^2(\mathbf{n})$; and $|\mathbf{A}_j(\mathbf{n})| = 1$, T indicates a transpose. Since, for any $f(\omega)$ and $\mathbf{e}$, we can express (11) as

$$\mathbf{U}(\mathbf{k}, \omega) = G(\mathbf{k}, \omega)f(\omega)\mathbf{e}, \quad (12)$$

we will restrict our further analysis to $G(\mathbf{k}, \omega)$. It is easy to see that this is the Green's tensor of wave equation (1).

Our next step is to reduce the triple inner integral in (9) to a double one. We combine (9) and (11) to obtain

$$G(\mathbf{x}, \omega) = \frac{1}{(2\pi)^3} \times \int_{R^3}\sum_{j=1}^{3}\frac{P_j(\mathbf{n})}{\mathbf{k}^2\nu_j^2(\mathbf{n}) - \omega^2}\exp[i|\mathbf{k}|(\mathbf{nx} + i\varepsilon)]d\mathbf{k}, \quad (13)$$

where $P_j(\mathbf{n}) = \mathbf{A}_j(\mathbf{n})\mathbf{A}_j^T(\mathbf{n})$ is the operator of projection on the direction of $\mathbf{A}_j$. Changing to the spherical coordinate system in (13) and introducing $s = |\mathbf{k}|$ and $d\Omega = \sin\theta d\theta d\phi$, we obtain the relation

$$G(\mathbf{x}, \omega) = \frac{1}{(2\pi)^3} \times \int_{|\mathbf{n}|=1} d\Omega \int_0^\infty \sum_{j=1}^{3}\frac{P_j(\mathbf{n})s^2}{s^2\nu_j^2(\mathbf{n}) - \omega^2}\exp[is(\mathbf{nx} + i\varepsilon)]ds, \quad (14)$$

where $\mathbf{n}^T = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. We express (14) as the sum of two terms, G_s and G_d, where

$$G_s = \frac{1}{(2\pi)^3} \times \int_{|\mathbf{n}|=1} d\Omega \int_0^\infty \sum_{j=1}^{3}\frac{P_j(\mathbf{n})}{\nu_j^2(\mathbf{n})}\exp[is(\mathbf{nx} + i\varepsilon)]ds, \quad (15)$$

$$G_d = \frac{1}{(2\pi)^3} \times \int_{|\mathbf{n}|=1} d\Omega \int_0^\infty \sum_{j=1}^{3}\frac{P_j(\mathbf{n})}{\nu_j^2(\mathbf{n})}\frac{\omega^2}{s^2\nu_j^2(\mathbf{n}) - \omega^2}\exp[is(\mathbf{nx} + i\varepsilon)]ds \quad (16)$$

and calculate the inner integrals with respect to s.

To calculate (15), we invoke the relation

$$\int_0^\infty \exp[is(\mathbf{nx} + i\varepsilon)]ds = \frac{1}{\varepsilon - i(\mathbf{nx})}. \quad (17)$$

Using (17) and taking into account the fact that $\nu_j(\mathbf{n})$ and $P_j(\mathbf{n})$ are independent of the direction of the normal vector, $\nu_j(-\mathbf{n}) = \nu_j(\mathbf{n})$ and $P_j(-\mathbf{n}) = P_j(\mathbf{n})$, we note that the imaginary part of (15) vanishes. We have

$$G_s = \frac{1}{(2\pi)^3}\int_{|\mathbf{n}|=1} d\Omega\frac{\varepsilon}{(\mathbf{nx})^2 + \varepsilon^2}\sum_{j=1}^{3}\frac{P_j(\mathbf{n})}{\nu_j^2(\mathbf{n})}. \quad (18)$$

Letting in (18) $\varepsilon \longrightarrow 0$, and using the well-known relation

$$\lim_{\varepsilon\to 0}\frac{\varepsilon}{\varepsilon^2 + z^2} = \pi\delta(z),$$

where $\delta(z)$ is the one-dimensional delta function, we obtain

$$G_s = \frac{1}{8\pi^2}\int_{|\mathbf{n}|=1} d\Omega\delta(\mathbf{nx})\Lambda^{-1}(\mathbf{n}). \quad (19)$$

Now, we consider (16). Since the inner integral in (16) uniformly converges with respect to the parameter $\varepsilon \geq 0$, we can calculate the integrand in the limit $\varepsilon \longrightarrow 0$. Next, we consecutively change the variables,

$\mathbf{n} \longrightarrow -\mathbf{n}$ and $s \longrightarrow -s$, to transform (16) into

$$G_d = \frac{1}{16\pi^3} \int\limits_{|\mathbf{n}|=1} d\Omega \sum_{j=1}^{3} \frac{P_j(\mathbf{n})}{v_j^2(\mathbf{n})} \times \int\limits_{-\infty}^{+\infty} \frac{\omega^2}{s^2 v_j^2(\mathbf{n}) - \omega^2} \exp[is(\mathbf{nx})] ds. \quad (20)$$

We use the residue theorem to calculate integrals of the form

$$I_j = \int\limits_{-\infty}^{+\infty} \frac{\exp[is(\mathbf{nx})]}{s^2 v_j^2(\mathbf{n}) - \omega^2} ds. \quad (21)$$

The integrand in (21) has poles at $s = \pm\omega/v_j$. According to the Jordan lemma, if $\mathbf{nx} > 0$ ($\mathbf{nx} < 0$), we can close the contour in the upper (lower) half-plane of (s). In either case, since $\mathrm{Im}\,\omega > 0$, one singular point $s = \omega/v_j$ exists inside the contour ($s = -\omega/v_j$). As a result, we obtain

$$I_j = \frac{i\pi}{\omega v_j} \exp\left[i\frac{\omega}{v_j}\mathbf{nx}\right] \quad (22)$$

for $\mathbf{nx} > 0$ and

$$I_j = \frac{i\pi}{\omega v_j} \exp\left[-i\frac{\omega}{v_j}\mathbf{nx}\right] \quad (23)$$

for $\mathbf{nx} < 0$. Here, we used the fact that integrals along the upper and lower half-circles are taken in the positive and negative directions, respectively. Unifying (22) and (23), we obtain

$$I_j = \frac{i\pi}{\omega v_j} \exp\left[i\frac{\omega}{v_j}|\mathbf{nx}|\right]. \quad (24)$$

As a result, expression (16) is transformed into

$$G_d = \frac{i\omega}{16\pi^2} \int\limits_{|\mathbf{n}|=1} d\Omega \sum_{j=1}^{3} \frac{P_j(\mathbf{n})}{v_j^3(\mathbf{n})} \exp\left[i\frac{\omega}{v_j}|\mathbf{nx}|\right]. \quad (25)$$

After the transformation, the Green's tensor is expressed as

$$G(\mathbf{x}, \omega) = \frac{1}{8\pi^2} \int\limits_{|\mathbf{n}|=1} d\Omega \delta(\mathbf{nx}) \Lambda^{-1}(\mathbf{n}) + \frac{i\omega}{16\pi^2} \int\limits_{|\mathbf{n}|=1} d\Omega \sum_{j=1}^{3} \frac{P_j(\mathbf{n})}{v_j^2(\mathbf{n})} \exp\left[i\frac{\omega}{v_j}|\mathbf{nx}|\right]. \quad (26)$$

In the direction of the vector $\mathbf{e}$, the displacement field generated by a point source with an arbitrary time dependence $f(t)$ is given by

$$\mathbf{U}(\mathbf{x}, t) = \frac{f(t)}{8\pi^2} \int\limits_{|\mathbf{n}|=1} d\Omega \delta(\mathbf{nx}) \Lambda^{-1}(\mathbf{n}) \mathbf{e} - \frac{1}{16\pi^2} \int\limits_{|\mathbf{n}|=1} d\Omega \sum_{j=1}^{3} \frac{f'(t - |\mathbf{p}_j\mathbf{x}|) P_j(\mathbf{n})}{v_j^3(\mathbf{n})} \mathbf{e}. \quad (27)$$

Using the identity $1 = H(\mathbf{px}) + H(-\mathbf{px})$, where $H(t)$ is the Heaviside step function, we rewrite (26) and (27) as

$$G(\mathbf{x}, \omega) = \frac{1}{8\pi^2} \int\limits_{|\mathbf{n}|=1} d\Omega \delta(\mathbf{nx}) \Lambda^{-1}(\mathbf{n}) + \frac{i\omega}{8\pi^2} \int\limits_{(\mathbf{px})>0} d\Omega \sum_{j=1}^{3} \frac{P_j(\mathbf{n})}{v_j^3(\mathbf{n})} \exp[i\omega(\mathbf{px})], \quad (28)$$

$$\mathbf{U}(\mathbf{x}, t) = \frac{f(t)}{8\pi^2} \int\limits_{|\mathbf{n}|=1} d\Omega \delta(\mathbf{nx}) \Lambda^{-1}(\mathbf{n}) \mathbf{e} - \frac{1}{8\pi^2} \int\limits_{|\mathbf{n}|=1} d\Omega H(\mathbf{nx}) \sum_{j=1}^{3} \frac{f'(t - |\mathbf{p}_j\mathbf{x}|) P_j(\mathbf{n})}{v_j^3(\mathbf{n})} \mathbf{e}. \quad (29)$$

If $f(t) = H(t)$, we use (29) and replace $\delta(t - \mathbf{px})H(\mathbf{px})$ with $\delta(t - \mathbf{px})H(t)$ to obtain

$$\mathbf{U}_H(\mathbf{x}, t) = \frac{H(t)}{8\pi^2} \left\{ \int\limits_{|\mathbf{n}|=1} d\Omega \delta(\mathbf{nx}) \Lambda^{-1}(\mathbf{n}) - \int\limits_{|\mathbf{n}|=1} d\Omega \sum_{j=1}^{3} \frac{\delta(t - \mathbf{p}_j\mathbf{x}) P_j(\mathbf{n})}{v_j^3(\mathbf{n})} \right\} \mathbf{e}. \quad (30)$$

To obtain an analogous expression for $f(t) = \delta(t)$, we differentiate (30). By applying the relations

$$\delta(t - \mathbf{px})\delta(t) = \delta(\mathbf{px})\delta(t) = v\delta(\mathbf{nx})\delta(t)$$

and

$$\sum_{j=1}^{3} P_j(\mathbf{n})/v_j^2(\mathbf{n}) = \Lambda^{-1}(\mathbf{n}),$$

it is easy to show that the product of $\delta(t)$ and the brackets in (30) vanish. Then,

$$\mathbf{U}_\delta(x, t) = -\frac{H(t)}{8\pi^2} \frac{\partial}{\partial t} \int\limits_{|\mathbf{n}|=1} d\Omega \sum_{j=1}^{3} \frac{\delta(t - \mathbf{p}_j\mathbf{x}) P_j(\mathbf{n})}{v_j^3(\mathbf{n})} \mathbf{e}. \quad (31)$$

After some simplification, equation (28) in [7] yields expression (31).

3. CALCULATION OF THE STATIC PART OF THE GREEN'S TENSOR

For the sake of brevity, we will call the first term in (26) and (27) the static Green's tensor. Here, our objective is to reduce it to an easily calculable form. To this end, we introduce a new coordinate system $(\tilde{k}_1, \tilde{k}_2, \tilde{k}_3)$, rotating the original system (k_1, k_2, k_3) about the k_1 axis by an angle ψ such that

$$\sin\psi = \frac{x_3}{r}, \quad \cos\psi = \frac{x_2}{r}, \tag{32}$$

where (x_1, x_2, x_3) are the coordinates of vector $\mathbf{x}$, and $r = \sqrt{x_2^2 + x_3^2}$. The new coordinates of the normal vector $\tilde{\mathbf{n}}$ are related to its original coordinates by the equation

$$\tilde{\mathbf{n}} = W\mathbf{n}, \quad W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{pmatrix}. \tag{33}$$

Under the transformation given by (33), the matrix $\Lambda(\mathbf{n})$ is replaced with

$$\hat{\Lambda}(\tilde{\mathbf{n}}) = \Lambda(W^T\tilde{\mathbf{n}}), \tag{34}$$

and the scalar product $\mathbf{nx}$ assumes the form $(\tilde{n}_1 x_1 + \tilde{n}_2 r)$ or $\tilde{\mathbf{n}}\hat{\mathbf{x}}$, where $\hat{\mathbf{x}}^T = (x_1, r, 0)$. In the new coordinate system, the static Green's tensor has the form

$$G_s(\mathbf{x}, t) = \frac{1}{8\pi^2}\int_0^{2\pi} d\tilde{\phi}\int_0^{\pi}\sin\tilde{\theta}\,\delta(\tilde{\mathbf{n}}\hat{\mathbf{x}})\hat{\Lambda}^{-1}(\tilde{\mathbf{n}})d\tilde{\theta}, \tag{35}$$

where $\tilde{\mathbf{n}} = (\sin\tilde{\theta}\cos\tilde{\phi}, \sin\tilde{\theta}\sin\tilde{\phi}, \cos\tilde{\theta})^T$, and $\tilde{\theta}$ and $\tilde{\phi}$ are the polar coordinates associated with the system $(\tilde{k}_1, \tilde{k}_2, \tilde{k}_3)$. Henceforth, we drop the tilda. Changing to the variable $u = \cot\theta$, $du = -d\theta/\sin^2\theta$, we eliminate $\mathbf{n}$ by introducing the vector $\mathbf{m}^T = (\cos\phi, \sin\phi, u)$ or $\mathbf{m} = \mathbf{n}/\sin\theta$ and, using $\delta(ax) = \delta(x)/|a|$, $\Lambda^{-1}(\mathbf{n}) = \Lambda^{-1}(\mathbf{m})/\sin^2\theta$, $\nu_k(\mathbf{n}) = \sin\theta\nu_k$, and $P_k(\mathbf{n}) = P_k(\mathbf{m})$, we obtain

$$G_s(\mathbf{x}, t) = \frac{1}{8\pi^2}\int_0^{2\pi} d\phi\int_{-\infty}^{+\infty} du\,\delta(\mathbf{m}\hat{\mathbf{x}})\hat{\Lambda}^{-1}(\mathbf{m}). \tag{36}$$

Next, we set

$$x_1 = R\cos\phi_0, \quad r = R\sin\phi_0, \tag{37}$$

where $R = |x| = \sqrt{x_1^2 + r^2}$, to obtain

$$G_s(\mathbf{x}, t) = \frac{1}{8\pi^2}\int_0^{2\pi} d\phi \times \int_{-\infty}^{+\infty} du\,\delta[R\cos(\phi - \phi_0)]\hat{\Lambda}^{-1}\begin{pmatrix} \cos\phi \\ \sin\phi \\ u \end{pmatrix}. \tag{38}$$

The argument of the delta function in (38) vanishes at $\phi_{1,2} = \phi_0 \pm \frac{\pi}{2}$, so that $\sin\phi_{1,2} = \pm\cos\phi_0$, and $\cos\phi_{1,2} = \mp\sin\phi_0$. Calculating the integral with respect to ϕ and taking into account the fact that the elements of the matrix Λ are homogeneous quadratic forms, we write a final expression for the static term of the Green's tensor:

$$G_s(\mathbf{x}, t) = \frac{1}{4\pi^2 R}\int_{-\infty}^{+\infty} du\,\hat{\Lambda}^{-1}\begin{pmatrix} \sin\phi_0 \\ -\cos\phi_0 \\ u \end{pmatrix}. \tag{39}$$

Integral (39) can be calculated in terms of residues at the roots of the determinant of the matrix $\hat{\Lambda}\begin{pmatrix} \sin\phi_0 \\ -\cos\phi_0 \\ u \end{pmatrix}$ in (34).

4. AN ASYMPTOTIC REPRESENTATION FOR THE GREEN'S TENSOR

To evaluate the leading term in (28) for $|\omega||x| \longrightarrow \infty$, we apply the stationary phase method [9]. Since the integral in the second term of expression (28) is calculated over the surface of a hemisphere defined by the circumference $\mathbf{px} = 0$, then according to [9], in addition to the contribution of stationary points in the phase function $S(\mathbf{p}) = \mathbf{px}$, we should also take into account the contribution of the integration domain boundary. We show that the contribution of the boundary is canceled by the static term in (39). To do this, we transform the second term in (28), using operations similar to those specified by equations (32)–(37) (see Section 3). As a result, this term takes the form

$$G_d = \frac{i\omega}{8\pi^2}\int_{\phi_0-\frac{\pi}{2}}^{\phi_0+\frac{\pi}{2}} d\phi\int_{-\infty}^{+\infty} du \times \sum_{j=1}^{3}\frac{\hat{P}_j(\mathbf{m})}{\hat{\nu}_j^3(\mathbf{m})}\exp\left[\frac{i\omega R\cos(\phi - \phi_0)}{\hat{\nu}_j(\mathbf{m})}\right], \tag{40}$$

where ϕ, ϕ_0, and u are the variables that were introduced in Section 3; and $P(\mathbf{n})$ and $\nu(\mathbf{n})$ are transformed into $\hat{P}(\mathbf{m})$ and $\hat{\nu}(\mathbf{m})$ in the same manner as $\Lambda(\mathbf{n})$ is transformed into $\hat{\Lambda}(\mathbf{m})$ [see (34) and (36)]. Accordingly, the integral over a hemisphere is transformed into an integral over a semicylinder of unit radius: $\phi_0 - \frac{\pi}{2} < \phi < \phi_0 + \frac{\pi}{2}$ in the coordinates u and ϕ.

When the phase function $S(\chi) = S_0 = \text{const}$ at the boundary of the integration domain $\partial\Omega$, the principal contribution of the boundary to integrals of the form

$$F(\omega) = \int_{\Omega} \Phi(\chi)\exp[i\omega S(\chi)]d\chi \tag{41}$$

is given by the equation ([9], p. 141)

$$F(\omega, \partial\Omega) = \frac{\exp[i\omega S_0]}{i\omega}\int_{\partial\Omega} \frac{\partial S(\chi)}{\partial \nu}|\nabla S(\chi)|^{-2}\Phi(\chi)d\sigma, \tag{42}$$

where $d\sigma$ is the surface differential of the boundary $\partial\Omega$, and $\frac{\partial S(\chi)}{\partial \nu}$ is the directional derivative along the external normal to $\partial\Omega$. In our case, the phase function is $S(\phi, u) = \frac{R\cos(\phi - \phi_0)}{\hat{\nu}(\mathbf{m})}$, and the cylinder is bounded by two straight lines, which are parallel to the u axis and have the coordinates $\phi = \phi_0 \pm \frac{\pi}{2}$. Accordingly,

$$\begin{gathered} \left.\frac{\partial S}{\partial \nu}\right|_{\phi = \phi_0 \pm \frac{\pi}{2}} = \mp\left.\frac{\partial S}{\partial \phi}\right|_{\phi = \phi_0 \pm \frac{\pi}{2}} = R/\hat{\nu}(\mathbf{m}_{1,2}), \\ \left.|\nabla S|^2\right|_{\phi = \phi_0 \pm \frac{\pi}{2}} = \left.\left(\frac{\partial S}{\partial \phi}\right)^2\right|_{\phi = \phi_0 \pm \frac{\pi}{2}}, \end{gathered} \tag{43}$$

where $\mathbf{m}_{1,2} = (\mp\sin\phi_0, \pm\cos\phi_0, u)$. Using (41) and (42), we write the principal contribution of the boundary to (40) in the form

$$\begin{aligned} G_d(\omega, \partial\Omega) &= -\frac{1}{4\pi^2 R}\int_{-\infty}^{+\infty} du \sum_{j=1}^{3} \frac{\hat{P}_j(\mathbf{m}_{1,2})}{\hat{\nu}_j^2(\mathbf{m}_{1,2})} \\ &= -\frac{1}{4\pi^2 R}\int_{-\infty}^{+\infty} du \hat{\Lambda}^{-1}(\mathbf{m}_{1,2}), \end{aligned} \tag{44}$$

which cancels out (39). By changing the variable $u \longrightarrow -u$ and noting that $P(\mathbf{m})$ and $\nu(\mathbf{m})$ are independent of the sign of $\mathbf{m}$, it is easy to show that either $\mathbf{m}_1$ or $\mathbf{m}_2$ can be used in expression (44). Our conclusion concerning cancellation of the contribution of the boundary by the static term is consistent with the conclusion made in [5], where it was presented without proof.

We now seek the stationary points of the phase function $S(\mathbf{p})$. They are determined by the condition

$$\frac{\partial}{\partial \alpha}(\mathbf{px}) = \frac{1}{\nu(\mathbf{n})}\frac{\partial \mathbf{n}^T}{\partial \alpha}[\mathbf{I} - \boldsymbol{\xi}\mathbf{p}^T]\mathbf{x} = 0, \tag{45}$$

where $\alpha = \theta, \phi$. To derive (45), we used the relation

$$\frac{\partial \mathbf{p}}{\partial \alpha} = \frac{1}{\nu(\mathbf{n})}[\mathbf{I} - \mathbf{p}\boldsymbol{\xi}^T]\frac{\partial \mathbf{n}}{\partial \alpha}, \tag{46}$$

where $\xi_i = \frac{\partial \nu(\mathbf{n})}{\partial n_i}$ are the components of the wavefront velocity vector [10]. Moreover, the identity $\boldsymbol{\xi}\mathbf{p} = 1$ [10] entails the condition

$$\mathbf{n}^T[\mathbf{I} - \boldsymbol{\xi}\mathbf{p}^T]x = 0. \tag{47}$$

Combined with two conditions (45), condition (47) implies that the vector $[\mathbf{I} - \mathbf{p}\boldsymbol{\xi}^T]\mathbf{x}$ is orthogonal to three noncoplanar vectors, $\mathbf{n}$, $\frac{\partial \mathbf{n}}{\partial \theta}$, and $\frac{\partial \mathbf{n}}{\partial \phi}$, and, therefore, is a null vector. Thus, the stationary points of the phase function are determined by

$$[\mathbf{I} - \boldsymbol{\xi}\mathbf{p}^T]\mathbf{x} = 0 \quad \text{or} \quad \boldsymbol{\xi} = \frac{\mathbf{x}}{(\mathbf{px})}. \tag{48}$$

In geometrical terms, these are points on the refraction surface where vector $\boldsymbol{\xi}$ (perpendicular to that surface [10]) is parallel to vector $\mathbf{x}$ because (28) requires that $(\mathbf{px}) > 0$ at these points.

To obtain an asymptotic representation for the Green's tensor, we need expressions for $\det J$ and $\operatorname{sgn} J$, where J is the matrix of second derivatives of the phase function with respect to the coordinates θ and ϕ, and $\operatorname{sgn} J$ is the difference between the numbers of positive and negative eigenvalues of matrix J [9]. Now, we will show that these quantities can be reduced to simple expressions by rewriting them in terms of the principal curvatures of the refraction surface at the stationary point and the corresponding signum functions.

To do this, we make use of the identity

$$\boldsymbol{\xi}\frac{\partial \mathbf{p}}{\partial \alpha} = 0, \tag{49}$$

where, as above, $\alpha = \theta, \phi$. It is easy to verify that (49) is true by virtue of (46), combined with the identity $\boldsymbol{\xi}\mathbf{p} = 1$.

The elements of the matrix J are $\mathbf{x}\frac{\partial^2 \mathbf{p}}{\partial \alpha \partial \beta}$, where $\alpha, \beta = \theta, \phi$. Differentiating (49) divided by $|\boldsymbol{\xi}|$, then substituting the expression for $\boldsymbol{\xi}$ at a stationary point determined by (48), we arrive at the relation

$$\mathbf{x}\frac{\partial^2 \mathbf{p}}{\partial \alpha \partial \beta} = -|\mathbf{x}|\frac{\partial \mathbf{l}}{\partial \beta}\frac{\partial \mathbf{p}}{\partial \alpha}, \tag{50}$$

where the right-hand side contains the elements of matrix A_2 of the second fundamental form for the refraction surface (see [11]), multiplied by $|\mathbf{x}|$, where $\mathbf{l} = \boldsymbol{\xi}/|\boldsymbol{\xi}| = \mathbf{x}/|\mathbf{x}|$ is the normal to the refraction surface at the stationary point. The determinant of A_2 is given by [11]

$$\det A_2 = \left(\frac{\partial \mathbf{p}}{\partial \theta} \times \frac{\partial \mathbf{p}}{\partial \phi}\right)^2 k_r, \tag{51}$$

where $k_r = k_1 k_2$ is the total curvature of the refraction surface, and k_1 and k_2 are its principal curvatures. Combined with (46), this leads to

$$\left(\frac{\partial \mathbf{p}}{\partial \theta} \times \frac{\partial \mathbf{p}}{\partial \phi}\right)^2 = \frac{\sin\theta^2}{v^2}\xi^2. \tag{52}$$

Hence, we immediately obtain

$$\det J = \mathbf{x}^2 \det A_2 = \mathbf{x}^2 \xi^2 \frac{\sin\theta^2}{v^2} k_r, \tag{53}$$
$$\operatorname{sgn} J = \operatorname{sgn} A_2 = \operatorname{sgn} k_1 + \operatorname{sgn} k_2,$$

where

$$\operatorname{sgn} k = \begin{cases} 1, \text{ if } k > 0 \\ -1, \text{ if } k < 0. \end{cases}$$

Singular stationary points at which $k = 0$ should be treated by applying a different version of the stationary phase method, which is not considered here (see [5, 9]).

Numbering the stationary points $\breve{\mathbf{p}}_j^{(i)}$ of the jth refraction surface that satisfy (48) with the index $i = 1, \ldots, N_j$, we obtain the final expression for the asymptotic representation of the Green's tensor:

$$G(\mathbf{x}, \omega) = -\frac{i}{4\pi|\mathbf{x}|}\sum_{j=1}^{3}\sum_{i=1}^{N_j}\frac{P_j(\breve{\mathbf{p}}_j^{(i)})}{|\xi(\breve{\mathbf{p}}_j^{(i)})|\sqrt{k_r(\breve{\mathbf{p}}_j^{(i)})}} \tag{54}$$
$$\times \exp\left[i\frac{\pi}{4}(\operatorname{sgn} k_1(\breve{\mathbf{p}}_j^{(i)}) + \operatorname{sgn} k_2(\breve{\mathbf{p}}_j^{(i)}) + \omega \breve{\mathbf{p}}_j^{(i)}\mathbf{x})\right].$$

This expression agrees with the one obtained by Musgrave [12].

5. CONCLUSION

We have proposed a new method of deriving an integral representation for the vector of displacement induced in an anisotropic medium by a localized force with an arbitrary time dependence. The method involves two key steps.

First, the localized source is approximated by a spherically symmetrical function depending on a single parameter, which yields a point source proper as the parameter goes to zero. Accordingly, we can use a Fourier transform to construct a solution by the classical method. Note that the parameter ensures consistent regularization of the integrals obtained.

Second, in an intermediate integral representation of the solution, a change of variables is performed by making use of the central symmetry (with respect to the spatial frequencies) of the matrices contained in the integrand. This replacement allows for applying the residue theorem and reducing the multiplicity of the original integrals by one, without calculating the time-frequency integral. Next, a solution is readily obtained for a source with an arbitrary time dependence. The error made in [6] stems from the very fact that the time-frequency integral was calculated first, whereas the spectral source function was assumed to be arbitrary; i.e., its singularities were ignored.

Moreover, this paper presents a simple method of calculating the static Green's tensor, which substantially simplifies calculation of the leading term in the asymptotic expression for the required solution, obtained by the stationary phase method in the case of a nondegenerate stationary point.

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